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Allocating electricity transmission costs through tracing: a game-theoretic rationale[☆]

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Abstract

Tracing is a method of assigning flows in an electricity network to particular generators and loads, assuming perfect mixing at each node. It can be used to assign costs to transmission users. We show that the resulting allocation is equal to the Shapley value of an equivalent co-operative game.

Keywords: Electricity transmission; Cost allocation; Shapley value

1. Introduction

Around the world, as electricity industries are restructured and liberalized, electricity is becoming a commodity to be bought and sold by generators, suppliers and other traders. As vertically integrated utilities are broken up, end users and distributors are able to buy power from distant generators. No commodity can be traded, however, unless there are appropriate arrangements for its delivery. This is the responsibility of transmission companies, and the special nature of electricity poses some challenges. These make transmission charging a complex subject, and different approaches have been adopted around the world [4]. To allow electricity trades between systems with different charges it is necessary to agree protocols for cross-border trades. In particular, system operators need to know how much a given trade uses the network, in order to allocate an appropriate portion of their costs to that trade. This paper discusses a technique, tracing, to determine how much each of a number of trades uses different parts of the electricity network. We provide a rationale for the technique—it is the equilibrium of a co-operative game formulation of the transmission cost allocation problem.

Pricing electricity transmission along a single line is straightforward. The cost of losses can be calculated, and if the line is congested, the difference in the marginal cost of power at each end of the line gives the shadow cost of this congestion. A competitive auction of the right to use the line would set its price

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equal to this cost. With economies of scale the total cost of an optimally sized line would exceed the revenues from such an auction, but it is straightforward to identify the companies making use of the line, and charge them for these common costs.

In a meshed network, it is impossible to say which generator is supplying which load because power flows through every line in the network. The traditional "contract path" approach ignored this, simply defining a single route between two points, along which the power was deemed to flow, and paying only transmission owners along that route. This becomes increasingly inequitable as the volume of transactions, and of potential "loop flows", passing through lines remote from the contract path, increases. The nodal pricing approach sets prices based on the marginal cost of generation or of meeting an extra unit of demand at each node on the network, assuming an optimal dispatch. Transmission charges are defined as the cost of moving power between each location and a "swing bus". These charges will generally be insufficient to meet the total cost of the transmission network [8], and some additional "common costs" must be recovered.

When considering cross-border transmission, cost recovery becomes more complicated. Nodal pricing based on marginal costs from an optimal dispatch is unlikely to be implementable, although the right to use a congested cross-border interconnector is sometimes auctioned. We still need to recover the common costs of these interconnectors (over and above any auction revenues), and the costs of any system reinforcement within a country needed to accept the flows from the interconnector. The easy option is simply to recover the costs incurred by each country from electricity charges within that country. This may not be equitable, particularly for countries with a high proportion of transit flows.

The main problem in designing a system that allocates transmission costs on an international basis is assigning responsibility for the power flows, and a tracing-based methodology is suggested in [2] that does just this. The underlying assumption is that the flow of power in the network can be represented by a directed graph in which the flows mix proportionally at every node. Then the costs of loop flows could be allocated to the agents causing them. It would also allow transit charges to be related to the costs involved in each route. This is applied in [3] to cross-border flows.

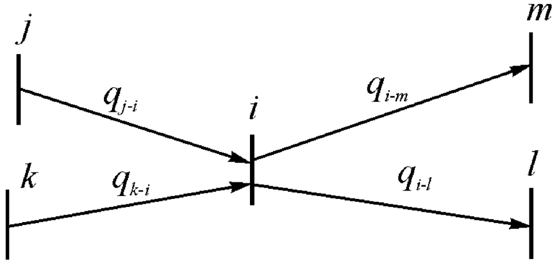

Fig. 1. Proportional sharing rule.

The detailed design of tariffs applying the tracing principle is not the focus of this paper. Instead, we wish to illustrate its theoretical link to the Shapley value, a well-known solution concept in cost allocation problems. In the next section we introduce tracing. Section 3 discusses the game-theoretic justification for the tracing methodology. Section 4 concludes.

2. The tracing methodology

2.1. The principle

Conventional wisdom is that it is impossible to trace the flow of power from individual generators to individual loads in meshed transmission networks. Assuming, however, that at any network node the inflows are distributed proportionally between the outflows, it is possible—by following the acyclic directed graph of flows in the network—to trace how real and reactive power flows in the network from individual sources to individual sinks. Transmission charges can then be calculated as in the traditional contract path approach but with the fundamental difference that the paths are not arbitrary. A tracing-based method overcomes problems related to charging based on marginal principles, while providing better signals than the postage stamp or contract path methods [2].

Electricity tracing is based on the proportional sharing rule illustrated in Fig. 1, which shows node i connected to two upstream nodes, j and k, and two downstream nodes, m and l, by four lines: $j-i$, $k-i$, $i-m$, and $i-l$. Real power flowing into node i is denoted by q_{j-i} and q_{k-i}, respectively, while power flowing out of node i is denoted by q_{i-m} and q_{i-l}; $q_{j-i}+q_{k-i}=q_{i-m}+q_{i-l}$. Nodes j and k (respectively, m and l) can be either some other nodes in the system or local generators (local demands) supplying (supplied from) node i. With no additional information, a logical assumption about how inflowing power is distributed among outflows, is that the network node is a perfect "mixer" of incoming flows so that nodal inflows are shared proportionally between the outflows. This implies:

- q_{i-m} is assumed to consist of two components: $[q_{j-i}/(q_{j-i}+q_{k-i})]q_{i-m}$ coming from q_{j-i} and $[q_{k-i}/(q_{j-i}+q_{k-i})]q_{i-m}$ coming from q_{k-i}.
- Similarly q_{i-l} is assumed to consist of $[q_{j-i}/(q_{j-i}+q_{k-i})]q_{i-l}$ coming from q_{j-i} and $[q_{k-i}/(q_{j-i}+q_{k-i})]q_{i-l}$ coming from q_{k-i}.
- q_{j-i} is assumed to consist of $[q_{i-m}/(q_{i-m}+q_{i-l})]q_{j-i}$ going to q_{i-m} and $[q_{i-l}/(q_{i-m}+q_{i-l})]q_{j-i}$ going to q_{i-l}.
- q_{k-i} is assumed to consist of $[q_{i-m}/(q_{i-m}+q_{i-l})]q_{k-i}$ going to q_{i-m} and $[q_{i-l}/(q_{i-m}+q_{i-l})]q_{k-i}$ going to q_{i-l}.

The proportional sharing principle can be extended to all the network nodes as shown in the next section.

2.2. Using the tracing-based methodology to allocate transmission charges

Tracing can be conducted either by graph-search algorithm [6] or by solving linear equations [2]. Here we present the latter as it deals easily with circular flows, which create cycles in the digraph of flows and prevent the use of the graph-search algorithm [3]. As a simplification, losses have been neglected but they can be easily included.

We start by assigning power flows to individual generators. The total power flow through a node (sum of nodal inflows or outflows) can be expressed, when looking at the inflows, as

$$q_i=\sum_{j\in\alpha_i^{(\mathrm{u})}}|q_{i-j}|+q_{\mathrm{G}i}\quad \text{for } i=1,2,\ldots,n, \tag{1}$$

where $\alpha_i^{(\mathrm{u})}$ is the set of upstream nodes supplying node i directly (i.e. power must flow towards node i in the relevant lines), q_{i-j} is the flow in line $i-j$ and $q_{\mathrm{G}i}$ is generation at node i. As losses have been neglected, $|q_{i-j}|=|q_{j-i}|$ and the flow q_{i-j} can be replaced by $(|q_{j-i}|/q_j)q_j$. Now Eq. (1) can be re-written as

$$q_i-\sum_{j\in\alpha_i^{(\mathrm{u})}}\frac{|q_{j-i}|}{q_j}q_j=q_{\mathrm{G}i}\quad\text{or}\quad \mathbf{A}_{\mathrm{u}}\mathbf{q}=\mathbf{q}_{\mathrm{G}}, \tag{2}$$

where $\mathbf{q}$ is the vector of nodal flows and $\mathbf{A}_{\mathrm{u}}$ is the upstream distribution matrix defined as

$$[\mathbf{A}_{\mathrm{u}}]_{ij}=\begin{cases}1 & \text{for } i=j,\\ -|q_{j-i}|/q_j & \text{for } j\in\alpha_i^{(\mathrm{u})},\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

As $\mathbf{q}=\mathbf{A}_{\mathrm{u}}^{-1}\mathbf{q}_{\mathrm{G}}$, the ith nodal flow can be calculated as

$$q_i=\sum_{k=1}^{n}[\mathbf{A}_{\mathrm{u}}^{-1}]_{ik}q_{\mathrm{G}k}. \tag{4}$$

Now the outflow from node i in line $i-l$ can be calculated as

$$|q_{i-l}|=\frac{|q_{i-l}|}{q_i}q_i=\frac{|q_{i-l}|}{q_i}\sum_{k=1}^{n}[\mathbf{A}_{\mathrm{u}}^{-1}]_{ik}q_{\mathrm{G}k}\quad\text{for all } l\in\alpha_i^{(\mathrm{d})}, \tag{5}$$

where α_i^{d} is the set of downstream nodes supplied directly from node i (that is power flows from those nodes to node i in the relevant lines) and n is the number of nodes in the system.

The above equation effectively breaks down line flow q_{i-l} into components due to individual generations $q_{\mathrm{G}k}$. Hence, it can be used to assign responsibility for using the line among all the generators in the system. The total cost can be obtained by summing up all the shares.

For the cost allocation to the loads, it can be shown in a similar way, [2], that

$$|q_{i-j}|=\frac{|q_{i-j}|}{q_i}\sum_{k=1}^{n}[\mathbf{A}_{\mathrm{d}}^{-1}]_{ik}q_{\mathrm{L}k}\quad\text{for all } j\in\alpha_i^{(\mathrm{u})}, \tag{6}$$

where $q_{\mathrm{L}k}$ is load demand at node k and $\mathbf{A}_{\mathrm{d}}$ is a downstream distribution matrix defined as

$$[\mathbf{A}_{\mathrm{d}}]_{il}=\begin{cases}1 & \text{for } i=l,\\ -|q_{l-i}|/q_l & \text{for } l\in\alpha_i^{(\mathrm{d})},\\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

Eq. (6) breaks down the line flow into components due to individual loads and hence it could be used to assign responsibility for costs incurred. A discussion of how charges could be set in this way is contained in [5].

In the next section, we argue that the cost allocation stemming from tracing is equivalent to that produced by the Shapley value of a co-operative game, and that tracing is therefore an appropriate way of allocating costs.

3. Game theoretic rationale of tracing

3.1. Cost allocation games

The problem is one of dividing the cost of a jointly used facility among participants in a co-operative venture. If we consider the allocation among generators, $i=1,2,\dots,n$ who use the transmission grid to supply their generated power, q_i, to users, the objective is to divide up the total costs $c(q_1,\dots,q_n)$ among the exporters in an efficiency-inducing and individually as well as jointly acceptable way. This cost allocation process can be carried out line by line, since total cost must sum up to the costs attributed to the lines. Because power flows are additive in any line, the cost function for any line takes the form: $c(\Sigma q_i)$. To proceed line by line, it is necessary to "trace", for each generator, the distributed flow of its power over all the lines.

Under marginal pricing, the impact of each generator on transmission loss in a line depends on power flows that are attributed to other generators on the line. The non-linearity of transmission loss and other costs on power flow vests a negative externality between generators. According to the marginal principle, incremental units of power flowing in the line cause greater loss. An exact allocation of the costs under the marginal principle requires that the contribution assigned any unit of power depends on the order in which it is assumed to increase the flow in the line. But there is no logical reason for accepting any one ordering of the flow of units of power, over any other, and so fairness requires a principle of symmetry.

The natural framework for the study of cost allocation problems is game theory [10]. Willing co-operation in the joint enterprise is the essence of cost sharing, and this is the focus of the theory of co-operative games, which provide a method for exact allocation of joint costs (or benefits) with fair treatment of the parties. The solution concept used below is due to [9], and the essential idea is that co-operative participation requires that each player is allocated what she can gain for herself, through membership in all possible coalitions of the set of players. The Shapley value solution concept has been used in a number of cost allocation models, for example, [7].

Game theoretic formulations define the game in terms of players, strategies and coalitions. In standard formulations, players exercise real choices in terms of strategies or coalitions. The game we define is an artificial game, intended to explain the logical justification for the proportional sharing principle. It is constructed to show that in a transmission loss allocation context, the proportional sharing assumption leads to a cost allocation solution that satisfies all the desirable properties one would look for in a solution.

3.2. The game

A general cost allocation game is fully specified in terms of the (finite) set of participants or players, $N := \{1,2,\dots,n\}$, their demands to supply through the grid, represented by the vector $q := (q_1,q_2,\dots,q_n)$ and $c = c(q_1,q_2,\dots,q_n)$, the cost function (here the transmission costs) or the characteristic function. This is the data of the allocation problem.

3.2.1. Players

The context of transmission cost allocation is that of a fixed number of generators supplying a set of lines. The levies to recover transmission costs must ultimately fall on generators, and it is natural to think of N as the set of generators. However, transmission costs arise for the flow of power, and it is more useful to identify the set of players as the set of units of power (e.g., MWs) flowing through the network. The cost allocation suggested by the equilibrium of this co-operative game would specify a levy for each player; i.e., each unit of power. The allocation of each generator can be obtained by adding up the levies upon the units of power generated by it. In the general cost allocation context, the characteristic function, c, specifies the minimal cost that will be incurred by each *coalition* of players arranging matters to suit its members. The notion of a coalition requires some interpretation to fit this context.

3.2.2. Coalitions

Coalitions capture the essential strategic element in co-operative games. Rational players may be expected to take advantage of possibilities of coalition formation. For example, in a stable equilibrium each participant will have compared any proposed allocation with what it is able to get by "working alone" to the extent that is possible. Further, any group of players who find that they can do better for themselves by co-operating only among themselves and excluding others from their arrangement, could form a coalition and hold out for their worth, in the formation of any other coalition. The equilibrium must respect all prospects of such coalition formation. It follows that the "worth" of any player, the share the player can be expected to get in the game as a whole, must be related to her worth to all possible coalitions. In the canonical formulation, each subset of $\{1,\dots,R\}$ is a potential coalition; there are 2^R coalitions. The characteristic function, c, attaches a real number, denoting the minimal cost that will be incurred by it, to each one of 2^R coalitions possible. If an allocation is such that none of all possible coalitions can do better for themselves, it is a good candidate for the equilibrium of the game. Such allocations are said to be in the core of the game, and denote solutions acceptable to all players. This general notion of the coalition can be interpreted to fit the problem of transmission loss allocation, as follows.

3.2.3. Shapley value

Consider first the network segment with only one outflow line, as in Fig. 2. How is the loss in line $i-m$ to be allocated to the upstream exporters j and k? To explicitly represent coalition formation in this context, it is useful to have a labelling system to refer to the players; for the moment, abstracting away the identity of the originating exporter. Consider a one-to-one map from the set of MWs flowing through the node to the set of natural numbers, running from 1 through R, where R is the total number of MWs in the nodal flow. The precise nature of this mapping does not matter. The numbers are labels and have no sequential interpretation relating to power flow. From a purely accounting point of view, we could consider the flow from the node to the outflow line as a process whereby players are treated as flowing into the line, one at a time, in the order in which they have been labelled, $1,\dots,R$.

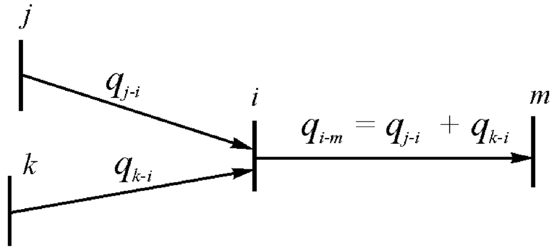

Fig. 2. Network segment with single outflow line.

We can proceed by constructing a co-operative game using the above framework. Let π denote one permutation of the set $\{1,\dots,R\}$, with the players accounted as flowing out in the sequence $\pi(1),\pi(2),\dots,\pi(R)$. Each $i\in\{1,\dots,R\}$, can be thought of as determining its worth relative to permutation π, based on the incremental cost when the accounting is done according to this order. The incremental, or marginal cost vector relating to permutation π is given by

$$m_i^\pi(c) \equiv c(P(\pi,i)\cup\{i\}) - c(P(\pi,i)), \tag{8}$$

where $P(.)$ denotes the set of "predecessors" of i with respect to $\pi, P(\pi,i) \equiv \{j\in\{1,\dots,R\} \mid \pi(j)<\pi(i)\}$. Incremental cost is increasing in $|P|$, the number of predecessors:

$$m_i^\pi(c) = r((|P+1|)^2 - (|P|)^2). \tag{9}$$

Obviously, i places highest value on that permutation where it is the first to be "accounted" to flow out, leaving it with the smallest incremental transmission loss allocation. It is obvious that with this allocation to each individual MW, the total cost would not be covered; this is not a feasible allocation.

One exact loss allocation rule is immediate. Each MW could be charged with the incremental transmission loss when it joins its "predecessors" (from an accounting point of view) in the outflow line. This serial cost-sharing rule recovers actual cost exactly. With a convex cost function, the incremental loss attached to a MW will be higher, the larger the number of its predecessors in the outflow. The cost recovery rule has the efficiency inducing marginal principle built into it, albeit in an unfair way: the charge depends critically on the order in which players are considered to enter the line, and this is based on the arbitrary labelling procedure. Can this procedure be modified to ensure fair treatment to all players?

Denote the set of all possible permutations of $\{1,\ldots,R\}$ by ΠR. Each $\pi\in\Pi R$ can be considered a coalition. This equivalence, in the context of this game, is based on the power of each coalition $S\subset\{1,\ldots,R\}$ to orchestrate a permutation π only to the extent that members of S are permuted. For each coalition $S\in 2^R$, the cost $c(S)$ of S is defined to be the minimum of the sum of the allocations of all the players in S, taken over all the permutations that can be orchestrated by S:

$$c(S)=\min_{\pi\in\Pi_R} m_c^{\pi}(S)\quad \text{for all } S\in 2R$$

$$\text{where } m_c^{\pi}(S)=\sum_{i\in S} m_i^{\pi}(c). \tag{10}$$

In the canonical formulation, the Shapley value allocation $\phi(c)=\{\phi_i(c)\}_{i=1,\ldots,R}$ is given by

$$\phi_i(c)=\sum_{S\subset\{1,\ldots,R\}\setminus\{i\}}\frac{|S|!(R-|S|)!}{R!}\times(c(S\cup\{i\})-c(S)). \tag{11}$$

When permutations are interpreted as coalitions, the Shapley value allocation is the average, taken over all permutations, of the marginal vectors of the game:

$$\phi(c)\equiv\left\{\frac{1}{R!}\sum_{\pi\in\Pi_R} m_i^{\pi}(c)\right\}_{i=1,\ldots,R}$$

$$=\left\{\frac{1}{R!}\sum_{\pi\in\Pi_R}(c(P(\pi,i)\cup\{i\})-c(P(\pi,i)))\right\}_{i=1,\ldots,R}. \tag{12}$$

This allocation will add up exactly to the total loss. Symmetry of the allocation arises from the equal consideration given to every possible ordering of the MWs. The Shapley value captures the idea that the worth of an individual player is the average of her worth in all possible coalitions. Each coalition is one permutation of the ordered set of player labels, denoting one possible order in which players can be *accounted* to have increased the flow in the line. From the point of view of a player, the set of permutations ΠR, signifies the set of all possible incremental (marginal) costs due to transmission loss, $m_{\cdot}^{\pi}(c);\pi\in\Pi_R$, that she could be potentially charged with. Each ordering gets the same weight $1/n!$; and the allocation for each MW is the average over $n!$ potential contributions to loss; the average over all possible marginal costs that can be attributed to it, the average cost per MW. This is the same for all players in this game.

The Shapley value is a suitable solution concept because it satisfies all the desirable properties we demand of a cost allocation rule. It lies in the core of the game; i.e., no coalition can do better, and so the allocation will be acceptable to all players. It is symmetric and fair, and also monotonic and additive. The monotonicity property guarantees that the charges will be non-negative, and the system will not lead to any player subsidizing another. The additivity property is useful if we were considering other types of charges, such as use of system charges, added on to transmission charges. It guarantees that if charges were decomposable into components, then the order in which the component-wise allocation is done will not make a difference to the cost allocation.

The power flow in the line, and the associated transmission cost, is attributable only on the basis of the total number of units of power flowing through the line, and not on their provenance. Under this solution concept, in the face of any symmetric cost function, fairness demands equal treatment of each MW, regardless of provenance. The transmission cost allocated to each MW of power flowing in the line is the same. Thus the loss allocation for each generator that supplies a single outflow line is proportional to the share of its generated output in that line. Proportional sharing follows directly from accepting the Shapley value as the solution concept.

3.2.4. Allocation of inflows to out flows

This logic also applies if there were more than one outflow line (for example, the network segment shown in Fig. 1). From an accounting point of view, one may consider the outflow from the node to the different lines to be toted up, MW by MW, in some order, for example, in the order in which the units of power have been labelled, 1 through R. Given the accounting procedure, the Shapley value allocation is based on equal consideration of all possible permutations of the set $\{1,\ldots,R\}$. If each $\pi \in \Pi_R$ has the same probability $(1/R!)$, this implies that each MW has equal probability of being allocated to any of the outflow lines. In other words, the proportional sharing rule is implicit in the determination of the Shapley value of this cost allocation game. Cost allocation over the whole network follows additively from cost allocation in each of the lines in the network. Thus the proportional sharing rule extends to the entire game. Further, the assumption that the set of players is a finite set—of units of power (e.g. MW)—can be relaxed. If player size goes to zero, the non-trivial generalization of the Shapley value to atomic games in [1] preserves the validity of the proportional sharing principle.

4. Conclusion

This paper analyzed the assumptions underlying a methodology for transmission pricing in a meshed network. We have argued elsewhere [5] that this methodology could be particularly applicable to recovering the common costs of inter-system trades; that is, those that are not recovered by marginal cost pricing schemes. The method establishes paths of real and reactive power flows from individual sources to individual sinks—following the directed graph of flows in the network. The implicit assumption is that, at any network node, the incoming flows are proportionally distributed among the outgoing flows. The aim of this paper was to critically examine this assumption, which can be neither proved nor disproved physically. We rationalized the principle using co-operative game theory, and showed that the Shapley value, the solution concept that has all the desirable properties one may demand of a cost allocation scheme, justifies the proportional sharing rule.

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