Proportional sharing assumption in tracing methodology

J.W. Bialek and P.A. Kattuman

Abstract: In order to overcome problems related to the marginal pricing of transmission costs, tracing methodology has been proposed as an alternative, most notably for transmission pricing of cross-border trades in Europe. The tracing methodology is based on the assumption that, at any network node, the incoming flows are proportionally distributed among the outcoming flows. This assumption can be neither proved nor disproved physically and the authors aim to rationalise it. The analysis presented here is of the loss allocation problem. First it is shown that the proportionality assumption leads to the cost allocation which is aggregation invariant. Then the proportional sharing principle is rationalised using game theory and the information theory. We show that the Shapley value solution concept, which satisfies all properties one may demand of a loss allocation scheme, substantiates the proportional sharing rule. We have also shown that the rule can be derived from the maximum entropy principle.

1 Introduction

Around the world, electricity industries are being restructured and liberalised. Electricity is now a commodity, bought and sold by generators, retailers (suppliers) and other traders. Vertically integrated utilities are being broken up, which allows end-users and distributors to buy power from more distant generators. No commodity can be traded, however, unless there are appropriate arrangements for its delivery. In the electricity industry, this is the responsibility of transmission companies, and the special nature of electricity poses a number of challenges. These make transmission charging a complex subject, and many different approaches have been adopted around the world.

Generally, the cost of transmission consists of four main elements: (i) transmission losses; (ii) operation and maintenance of the network; (iii) return on and depreciation of the capital equipment; (iv) the cost of resolving transmission constraints (congestion). This paper concentrates on the transmission loss allocation as an example of a transmission pricing methodology, but the considerations can be equally well applied to the other components too. The regulatory challenge is to devise an open access charging system that recovers common costs with fairness, while inducing efficient use of the grid by participants. The system must provide incentives for efficient use of the existing system as well as incentives for future development of the network, depending on the changing needs of the market. At the same time, the charging system must be practical enough for real-time application and transparent enough to be politically acceptable [1]. Different countries place different emphasis on each of these requirements and, as a result, there are hardly two countries in the world with identical transmission pricing regimes. This creates a problem whenever a trade is attempted which crosses a country border. One example of this is the interconnected network of USA and another is the interconnected UCTE network in Europe. In both cases, the interconnected network consists of a number of control areas (countries in Europe) with often incompatible internal transmission pricing regimes. This incompatibility is a serious impediment to the development of interchange transaction (or cross-border trades in Europe), especially when the trading parties are not in neighbouring countries (control areas). To overcome these problems, a recent report [2] suggested a possible use of two methodologies for transmission pricing of cross-border trades in Europe: marginal pricing, and tracing (also referred to as average participations or MW-tagging).

Marginal pricing for transmission is based on the theory of optimal pricing of electricity developed by Schweppe *et al.* [3]. In this approach, the optimal short-run price for electricity is determined at every node from the submitted bids (or cost curves) of every generator and every load. This price is a combination of the system marginal cost of generation at the swing bus, the impact on losses, and the impact on congestion. In a pool with true optimal nodal prices, there is no need to decompose the nodal price into generation and transmission because the transmission price is contained in the overall nodal price [4]. Many countries, however, require explicit pricing of transmission in which case the last two components of the nodal price (impact on losses and impact on congestion) may be treated as the short-run transmission price. The need for explicit transmission pricing is also apparent in the case of cross-border trades, when each country may have different internal trading arrangements.

Application of marginal pricing for transmission loss charging encounters a number of possible problems which are especially pronounced when developing a common transmission framework for an interconnected system. Firstly, calculation of marginal loss factors may be cumbersome for a large interconnected system. It requires much data exchange between the areas and a number of

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additional assumptions have to be made. Also the computational time for a system containing many thousands of nodes may be excessive. Secondly, the methodology itself is non-transparent and this opaqueness increases with the system size. Non-transparent and difficult to verify charges are bound to be met with suspicion from the market participants. Markets that are opaque will discourage new entry and lessen price competition. Thirdly, the marginal loss factors are known to be volatile, hence requiring of some additional hedging mechanisms. The cost of hedges increases the transaction costs. Fourthly, due to convexity of the loss function (transmission losses are approximately proportional to the square of the power flow), the sum of collected loss charges is higher than the actual sum of losses. Hence, an additional mechanism would be necessary in order to distribute the surplus. And finally, anomalies and opportunities for arbitrage could occur if marginal loss charging was applied for inter-area trades but the losses inside each area were not charged on the marginal basis. Making each area to adopt the same methodology, whether marginal or any other, would probably meet very strong resistance.

However, probably the main problem with applying the marginal loss factors would be the question of choosing the slack (or swing) node [5]. For example, when charging for losses, changing the position of the slack bus does not affect the relative values of marginal loss factors but shifts them all by the same amount, in effect changing the split of charges between the generators and the loads. Generally, a slack node located near a major load centre would tend to increase the total network charges paid by the generators, while a slack node located near a generation centre would increase the share of the demand in total payments. A change in the slack bus is then just a way of determining the global percentage of the network costs paid by all the consumers and all the producers. When the wholesale markets are perfectly competitive, any payment or tax charged on all the producers – such as a uniform adder to all transmission tariffs due to a specific location of the slack bus – would be passed on to the consumers via market prices. Thus the choice of the slack node would seem to be immaterial. Quite often however – and Europe is a good example of this – the market in a large interconnected system is far from perfectly competitive and hence the decision about the position of the slack node could have serious consequences for network users in individual systems (countries).

In order to overcome the problems with marginal pricing for transmission, [2] suggested an alternative in the form of tracing, or average participations, methodology [6,7]. Assuming that the inflows are distributed proportionally between the outflows at any network node, it is possible – by following the acyclic directed graph (digraph) of flows in the network – to trace how real and reactive power flows in the network, from individual sources to individual sinks. This paper tackles the main problem with the tracing methodology, i.e. its fundamental assumption that at any network node the inflows are distributed proportionally between the outflows [8]. The proportionality sharing assumption cannot be proved or disproved physically but it can be rationalised. First, we show that any other assumption would lead to the cost allocation which would not be aggregation invariant, i.e. traders would minimise their transmission charges by splitting into smaller units. Then we analyse the proportionality assumption using cooperative game theory and information theory. We conclude that the Shapley value, the solution concept which satisfies all properties one may demand of a loss allocation scheme, substantiates the proportional sharing rule. The rule can also be derived from the maximum entropy principle.

2 The tracing methodology

2.1 The principle

Conventional wisdom is that it is impossible to trace the flow of power from individual generators to individual loads in meshed transmission networks. Assuming however, that at any network node the inflows are distributed proportionally between the outflows, it is possible – by following the acyclic directed graph (digraph) of flows in the network – to trace how real and reactive power flows in the network, from individual sources to individual sinks [6, 9]. In other words, the tracing methodology allows us to 'tag MWs', i.e. establish physical paths linking generators and the loads. Consequently, transmission charges can be calculated, as with the traditional contract path approach. It should be stressed that the tracing paths are based on the physical power flows in the network while the contract paths are based on the financial contracts which usually do not reflect the physics of transmission.

Electricity tracing is based on the proportional sharing rule illustrated in Fig. 1 which shows node i connected with two upstream nodes, j and k, and two downstream nodes, m and l, by four lines: j–i, k–i, i–m, and i–l. Real power flowing into node i is denoted by q_j and q_k, respectively, while power flowing out of node i is denoted by q_m and q_l, respectively. Obviously $q_j + q_k = q_m + q_l$. Nodes j and k can be either some other nodes in the system or local generators supplying node i. Nodes m and l can be either some other nodes in the system or local demands supplied from node i. The question is, how the inflowing power is deemed to be distributed among the outflows? In the absence of any additional information, the most logical assumption is that the network node is a perfect 'mixer' of incoming flows so that nodal inflows are shared proportionally between the outflows. This would give the following result:

- q_m is assumed to consist of two components: $\frac{q_j}{q_j+q_k} q_m$ coming from q_j and $\frac{q_k}{q_j+q_k} q_m$ coming from q_k
- similarly q_l is assumed to consist of $\frac{q_j}{q_j+q_k} q_l$ coming from q_j and $\frac{q_k}{q_j+q_k} q_l$ coming from q_k
- q_j is assumed to consist of $\frac{q_m}{q_m+q_l} q_j$ going to q_m and $\frac{q_l}{q_m+q_l} q_j$ going to q_l
- q_k is assumed to consist of $\frac{q_m}{q_m+q_l} q_k$ going to q_m and $\frac{q_l}{q_m+q_l} q_k$ going to q_l

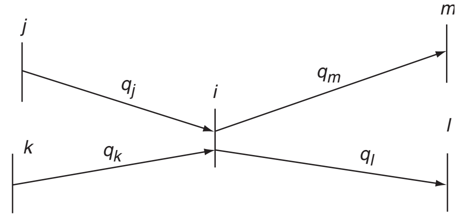

Fig. 1 *Proportional sharing rule*

The proportional sharing principle can be extended to all the network nodes and allows electricity to be traced in the network by a series of recursive calculations [9] Alternatively, it can be used to derive to form the tracing problem in a matrix form [6]. Both approaches can be proved to be computationally equivalent [10]. Although the graph-based approach is intuitively easier to understand, the matrix approach discussed below is easier to program and it allows circular flows which create cycles in the directed graph of network flows to be dealt with more easily.

2.2 Tracing-based loss allocation algorithm

The derivation of the tracing algorithm presented below is a modified version of the original derivation introduced in [6, 7]. This modification is simpler and allows the transmission loss charges to be calculated more directly.

Consider first the loss allocation to the loads illustrated in Fig. 2. Line $j{-}i$ connects a sending node j with a receiving node i. Both nodes may be connected to the rest of the system by a number of lines. Real power at the sending and receiving node is denoted as q_{j-i} and q_{i-j}, respectively. Real power demand at nodes j and i is q_{Dj} and q_{Di}, respectively, while real power generation at nodes j and i is q_{Gj} and q_{Gi}, respectively. The actual transmission loss in line $j{-}i$ is the difference between the sending and receiving end flows: $\Delta q_{j-i} = |q_{j-i}| - |q_{i-j}|$. Conceptually, the line also carries an unknown loss $\Delta q_{j-i}^{(u)}$ accumulated in all the lines upstream from node j in the digraph of flows. This upstream line loss can be calculated if we know the upstream nodal loss $\Delta q_j^{(u)}$ allocated to the sending node j. Assuming that $\Delta q_j^{(u)}$ is passed on proportionally to all the outflows from node j, $\Delta q_{j-i}^{(u)}$ and the loss allocated to the demand at node j can be calculated as

$$\Delta q_{j-i}^{(u)} = \frac{|q_{j-i}|}{q_j} \Delta q_j^{(u)} \text{ and} \quad \Delta q_{Dj} = \frac{q_{Dj}}{q_j} \Delta q_j^{(u)} \tag{1}$$

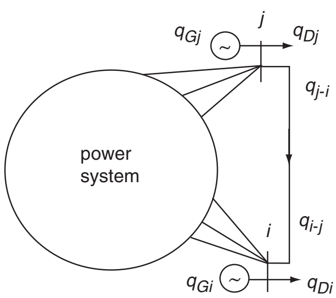

Fig. 2 *Tracing-based loss allocation*

where q_j is the total flow through node j (i.e. the sum of inflows or outflows), q_{Dj} is demand at node j and Δq_{Dj} is the loss allocated to that demand.

Summing up the actual transmission losses and the passed-on upstream losses in all the lines supplying node i gives the loss allocated to node i

$$\Delta q_i^{(u)} = \sum_{j \in \alpha_i^{(u)}} \Delta q_{j-i} + \sum_{j \in \alpha_i^{(u)}} \frac{|q_{j-i}|}{q_j} \Delta q_j^{(u)} \tag{2}$$

where $\alpha_i^{(u)}$ is the set of nodes supplying directly node i. Moving the second component on the right-hand side in (2) to the left-hand side allows us to express (2) in the matrix form

$$\mathbf{A}_u \Delta \mathbf{q}^{(u)} = \Delta \mathbf{q}_\Sigma^{(u)} \tag{3}$$

where $\Delta \mathbf{q}^{(u)}$ is the vector of unknown upstream nodal losses $\Delta q_j^{(u)}$ ($j = 1, 2 \ldots n$ where n is the number of nodes), $\Delta q_\Sigma^{(u)}$ is a vector with its ith element equal to $\sum_{j \in \alpha_i^{(d)}} \Delta q_{i-j}$ (i.e. sum of actual transmission losses in all the lines supplying node i) and $\mathbf{A}_u$ is the upstream distribution matrix defined as:

$$[\mathbf{A}_u]_{ij} = \begin{cases} 1 & \text{for } i = j, \\ -|q_{j-i}|/q_j & \text{for } j \in \alpha_i^{(u)} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Solving (3) gives the unknown vector $\Delta \mathbf{q}^{(u)}$ and the final allocation of the total transmission loss to individual loads is obtained from the second equation of (1).

For the loss allocation to the generators, let $\Delta q_i^{(d)}$ be the unknown transmission loss allocated to node i which has been accumulated downstream from node i in the digraph of network flows. Using similar reasoning as before, $\Delta q_j^{(d)}$ at the supplying node j can be calculated as the sum of actual transmission losses in all the lines supplied from j plus a proportion of all the downstream nodal loss allocations $\Delta q_i^{(d)}$

$$\Delta q_j^{(d)} = \sum_{j \in \alpha_i^{(d)}} \Delta q_{j-i} + \sum_{j \in \alpha_i^{(d)}} \frac{|q_{i-j}|}{q_i} \Delta q_i^{(d)} \tag{5}$$

where $\alpha_i^{(d)}$ is the set of nodes supplying directly node i (i.e. power must flow to node i in the relevant lines).

Moving the second component on the right-hand side of (5) to the left-hand side allows us to express (5) in the matrix form:

$$\mathbf{A}_d \Delta \mathbf{q}^{(d)} = \Delta \mathbf{q}_\Sigma^{(d)} \tag{6}$$

where $\Delta \mathbf{q}^{(d)}$ is the vector of all $\Delta q_i^{(d)}$ for $j = 1, 2 \ldots n$, $\Delta \mathbf{q}_\Sigma^{(d)}$ is a vector with its ith element equal to the sum of losses in all the lines supplied directly from node i, and $\mathbf{A}_d$ is the downstream distribution matrix with elements:

$$[\mathbf{A}_d]_{ji} = \begin{cases} 1 & \text{for } i = j, \\ -|q_{i-j}|/q_j & \text{for } j \in \alpha_j^{(d)} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Now Δq_{Gi}, the loss allocated to generator i, can be calculated using the proportional sharing principle as:

$$\Delta q_{Gi} = \frac{q_{Gi}}{q_i} \Delta q_i^{(d)} \tag{8}$$

where q_{Gi} is generation at node i.

3 Proportional or not proportional

The proportional sharing assumption may be intuitively accepted for the allocation of costs which are proportional to the flow value (like, for example, usage of transmission lines) but it may questioned whether it is appropriate for the allocation of cost components which are non-proportional to the flow value, like, for example, transmission losses. Indeed, [6] contains a version of the loss allocation algorithm in which transmission losses are allocated proportionally to the square of flows on the basis that the losses are proportional to the square of flows too. To investigate the assumption further, let us assume that the loss due to flow q_j in branch $j-i$ in Fig. 1 is Δq_j. If this loss is passed on proportionally to the outgoing flows, the loss passed on to q_m and q_l would be equal to

$$A_m^p = \Delta q_j \frac{q_m}{q_m + q_l} \quad A_l^p = \Delta q_j \frac{q_l}{q_m + q_l} \tag{9}$$

respectively, where superscript 'p' in A_m^p stands for proportional.

On the other hand, if the loss is passed on proportionally to the square of the flows, the loss passed on to q_m and q_l would be equal to

$$A_j^s = \Delta q_j \frac{q_m^2}{q_m^2 + q_l^2} \quad A_l^s = \Delta q_j \frac{q_l^2}{q_m^2 + q_l^2} \tag{10}$$

respectively, where superscript 's' stands for square.

Let us now analyse the consequences of the two rival loss allocation principles. Cost allocation proportional to the square of the flow may seem to reflect better the nature of losses but it fails one of the fundamental principles of cost allocation in that it is not aggregation invariant [11]. Cost allocation is allocation invariant, or consistent, if splitting a product into several equivalent products does not effectively change their prices. To prove that non-proportional sharing of costs fails that principle let us assume that flow q_m represents demand by a large user. The user, in order to minimise loss charges, can split himself into two, formally independent units, A and B, consuming q_{mA} and q_{mB}, repectively. Obviously $q_m = q_{mA} + q_{mB}$.

If proportional cost allocation is used, the total loss charge faced by the user will be

$$A_m^p = \Delta q_j \left(\frac{q_{mA}}{q_m + q_l} + \frac{q_{mB}}{q_m + q_l} \right) = \Delta q_j \frac{q_m}{q_m + q_l} \tag{11}$$

which is the same as in (9).

If the square cost allocation is used, the total loss charge faced by the user will be:

$$\begin{aligned} A_m^s &= \Delta q_j \left(\frac{q_{mA}^2}{q_{mA}^2 + q_{mB}^2 + q_l^2} + \frac{q_{mB}^2}{q_{mA}^2 + q_{mB}^2 + q_l^2} \right) \\ &= \Delta q_j \left(\frac{q_m^2 - 2q_{mA}q_{mB}}{q_m^2 + q_l^2 - 2q_{mA}q_{mB}} \right) \end{aligned} \tag{12}$$

It is easy to prove that $A_m^p > A_m^s$ so the non-proportional loss allocation would encourage users to divide themselves into smaller units. This shows that in order to prevent minimising loss charges by artificially splitting generation or demand, the proportional sharing principle must be used.

4 Game theoretic rationale of tracing

4.1 Cost allocation games

The principle behind the tracing methodology is not a physical principle, and we need to examine its rationale. The game theoretic analysis here deals only with the issue of transmission loss allocation but the results are equally applicable to the allocation of other transmission costs. A transmission network connects a certain number of nodes where electricity is either generated and/or consumed, through branches or lines. What is the best way to allocate the total transmission loss in the network between generators or the loads? In economics, problems of this type are usefully stated in the form of dividing the cost of a jointly used facility among participants in a co-operative venture.

We may start by considering cost allocation among generators, $i = 1, 2, \ldots, n$ who use the transmission grid to supply their generated power, q_i to users. It is necessary to divide up the total transmission loss $c(q_l, \ldots, q_n)$ among them. This cost allocation must be fair, and individually, as well as jointly acceptable. How can the contribution of each generator to the total transmission loss in the grid be determined? The task is to disentangle the contributions of the different generators. This allocation process can be carried out line-by-line, since total transmission loss is the sum of transmission losses in the lines. If it is possible to 'trace', for each generator, the distributed flow of its power over all the lines, then because power flows are additive in any line, the cost function for any line takes a simple homogeneous form: $c(q_1, \ldots, q_n) = c(\Sigma q_i) = r(\Sigma q_i)^2$, where r is resistance and i indexes the set of supplying generators. The cost function is strictly quadratic when q represents the line current. When power flows are considered, the relationship is approximately quadratic.

The essence of the marginal pricing rule applied to any convex cost function is that incremental units of power flowing in the line are deemed to cause greater loss. An exact allocation of the loss then requires that the contribution assigned to any unit of power depends on the order in which it is assumed to increase the flow in the line. With a homogeneous cost function there is no reason for accepting any one ordering of the flow of units of power over any other. Fairness requires that a principle of symmetric treatment be invoked.

A central focus of the theory of co-operative games with transferable utility is the equitable and exact division of a joint cost among economic agents who contribute to those costs [11, 12]. Willing co-operation in the allocation scheme is the essence of cost sharing, and this is the focus of the theory of co-operative games. The solution concept we use below, due to Shapley [13], has been popular in cost allocation problems. The essential idea is that co-operative participation requires that each player is allocated what can be gained by that player, through membership in all possible coalitions of the set of players.

Game-theoretic formulations define the game in terms of players, strategies and coalitions. In ordinary formulations, players exercise real choices in terms of strategies or coalitions. The game we define is an artificial game, intended to explain the logical justification for the proportional sharing principle. It is not a real game, but one constructed to show that in an transmission loss allocation context, the proportional sharing assumption leads to a cost allocation solution that satisfies all desirable properties one would look for in a solution.

4.2 The game

The key task is to define the appropriate game. It is relatively simple to choose and apply the appropriate solution concept from the set of well understood solution concepts to any game, once defined. A general cost allocation game is fully specified in terms of the (finite) set of participants or players, $N := \{1, 2, \ldots, n\}$, their demands to supply through the grid represented by the vector $q := (q_1, q_2, \ldots, q_n)$ and $c = c(q_1, q_2, \ldots, q_n)$ the cost function (here the transmission loss) or the characteristic function. This is the data of the allocation problem.

4.2.1 Players: The context of transmission loss allocation is that of a fixed number of generators supplying a set of lines. The levies to recover transmission losses must ultimately fall on generators, and it is natural to think of N as the set of generators. However, transmission loss is due to the flow of power, and it is more useful to identify the set of players as the set of units of power (e.g. MWs) flowing through the network. The cost allocation suggested by the equilibrium of this co-operative game would specify a levy for each player, i.e. each unit of power. The allocation of each generator can be obtained by adding up the levies upon the units of power generated by it. In the general cost allocation context, the characteristic (cost) function c, specifies the minimal cost that will be incurred by each coalition of players arranging matters to suit its members. The notion of a coalition requires some interpretation to fit this context.

4.2.2 Coalitions: Coalitions capture the strategic element in co-operative games. Rational players may be expected to take advantage of possibilities of coalition formation. For example, in a stable equilibrium each participant will have compared any proposed allocation with what it is able to get by 'working alone', to the extent that is possible. Further, any group of players who find that they can do better for themselves by co-operating only among themselves and excluding others from their arrangement, could form a coalition and hold out for their worth, in the formation of any larger coalition. The equilibrium must respect all prospects of such coalition formation. It follows that the 'worth' of any player, the share the player can be expected to get in the game as a whole, must be related to the player's worth to all possible coalitions. In the canonical formulation, each subset of $\{1, \ldots, R\}$ is a potential coalition; there are 2^R coalitions. The characteristic function c, attaches a real number denoting the minimal cost that will be incurred by it, to each one of 2^R possible coalitions. If an allocation is such that none of all possible coalitions can do better for itself, it is a good candidate for the equilibrium of the game. Such allocations are said to be in the core of the game and denote solutions acceptable to all players. This general notion of the coalition can be interpreted to fit the problem of transmission loss allocation.

4.2.3 Shapley value: Consider first the network segment with only one outflow line, as illustrated in Fig. 3. How is the loss in line i–m to be allocated to the upstream nodes j and k (they may be thought of as generators)? To explicitly represent coalition formation in this context, it is useful to have a labelling system to refer to the players; for the moment, abstracting away the identity of the originating generator. Consider a one-to-one map from the set of MWs flowing through the node to the set of natural numbers, running from 1 through R, where R is the total number of MWs in the nodal flow. The precise nature of this mapping does not matter. The numbers are labels and have no relationship to the flow of power, but from a purely accounting point of view, we could consider the flow from the node to the outflow line as a process whereby players are treated as fed to the line, one at a time, in the order in which they have been labelled, 1, …, R.

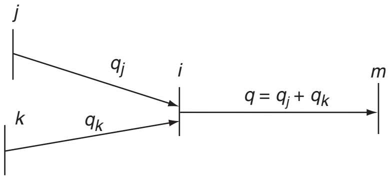

Fig. 3 *Network segment with single outflow line*

We can proceed by constructing a co-operative game using the above framework. Let π denote one permutation of the set $\{1, \ldots, R\}$, indicating a specific sequence denoted $\{\pi(1), \pi(2), \ldots, \pi(R)\}$. Each $i \in \{1, \ldots, R\}$, can be thought of as determining its worth in the permutation π, based on the incremental cost when the accounting is done according to this order in the flow. For each i, let $P(\pi, i)$ denote the set of 'predecessors' of i with respect to π, i.e. $P(\pi, i) \equiv \{j \in \{1, \ldots, R\} | \pi(j) < \pi(i)\}$. Then it is evident that for each $i \in \{1, \ldots, R\}$, the incremental or marginal cost $m_i^\pi(c)$, relating to permutation π is given by difference in costs with and without i:

$$m_i^\pi(c) \equiv c(P(\pi, i) \cup \{i\}) - c(P(\pi, i)) \qquad (13)$$

If the number of predecessors of i in permutation π is denoted by $|P(\pi, i)|$, then the incremental cost allocated to i according to π is increasing in $|P(\pi, i)|$. From the cost function, $c(\Sigma q_i) = r(\Sigma q_i)^2$:

$$m_i^\pi(c) = r((|P(\pi, i) + 1|)^2 - (|P(\pi, i)|)^2) \qquad (14)$$

Obviously, i places highest value on that permutation where it is the first to be 'accounted' to flow out, leaving it with the smallest incremental transmission loss allocation. It is obvious that with this allocation to each individual MW, the total cost would not be covered; this is not a feasible allocation.

One exact, but arbitrary, loss allocation rule presents itself immediately. Each MW could be charged with the incremental transmission loss when it joins its 'predecessors' (from the above accounting point of view) in the outflow line. Given π, the cost-sharing rules (13) and (14) above recover actual cost exactly. As noted before, with a convex cost function, the incremental loss attached to a MW will be higher, the larger the number of its predecessors in the outflow. The cost recovery rule has the efficiency-inducing marginal principle built into it, albeit in an unfair way: the charge depends critically on the order in which players are considered to enter the line, and this is based on the arbitrary labelling procedure. To modify this procedure and ensure fair treatment to all players we turn to the notion of coalitions.

Each subset S of $\{1, \ldots, R\}$ is a coalition. For each coalition S, the cost $c(S)$ is defined to be the minimum of the sum of the allocations of all the players in S. From the point of view of representing coalitions in terms of permutations of $\{1, \ldots, R\}$, each S can be viewed as having the power to orchestrate permutations of $\{1, \ldots, R\}$ where only members of S are permuted. From this point of view, $c(S)$ is the minimum of the sum of the allocations of all the players in S, taken over all the permutations that can be orchestrated by S. Formally, for a particular permutation π, the cost of coalition S can be defined as $m_c^\pi(S) = \sum_{i \in S} m_i^\pi(c)$. If we denote the set of all possible permutations of $\{1, \ldots, R\}$ by Π_R, then the cost allocation of coalition S is:

$$c(S) = \min_{\pi \in \Pi_R} m_c^\pi(S), \text{ for all } S \subset \{1, \ldots, R\} \qquad (15)$$

The Shapley value solution concept captures the idea that the worth of an individual player is the average of the player's worth in all possible coalitions. In the present context this is the average incremental cost attributable to the player, with the average taken over all possible coalitions. Consideration of all possible coalitions amounts to consideration of all permutations of set $\{1, \ldots, R\}$ that its 2^R subsets can 'orchestrate'. From this point of view, the set of all permutations, Π_R, signifies the set of all possible incremental (marginal) costs that a unit of power could be potentially charged with. In the Shapley value allocation, each ordering in Π_R gets the same weight 1/R! and the allocation for each MW is the average over R! potential contributions to loss. This average over all possible marginal costs that can be attributed to a player is the average cost per MW – the same for all players in this game.

$$\begin{aligned}\phi(c) \equiv \{\phi_i(c)\}_{i=1,\ldots,R} &\equiv \left\{\frac{1}{R!}\sum_{\pi\in\Pi_R} m_i^{\pi}(c)\right\}_{i=1,\ldots,R} \\ &\equiv \left\{\frac{1}{R!}\sum_{\pi\in\Pi_R}(c(P(\pi,i)\cup\{i\}) - c(P(\pi,i)))\right\}_{i=1,\ldots,R}\end{aligned} \tag{16}$$

The Shapley value is a suitable solution concept because it satisfies all the desirable properties we may demand of a cost allocation rule. This allocation will add up exactly to the total loss. The symmetric nature of the allocation arises from the equal consideration given to every possible ordering of the MWs. It lies in the core of the game; the implication is that no coalition can do better, and so the Shapley value allocation will be acceptable to all players. It is also monotonic and additive. The monotonicity property guarantees that the charges will be non-negative, and the system will not lead to any player subsidising another. The additivity property is useful if we were considering other types of charges, such as use of system charges, added on to transmission charges. It guarantees that if charges were decomposable into components, then the order in which the component-wise allocation is done will not make a difference to the cost allocation.

To summarise the argument of this section, power flow in the line and the associated transmission loss, depends only on the total number of units of power flowing through the line and not on their provenance or identity. Under this solution concept, in the face of any symmetric cost function, fairness demands equal treatment of each MW, regardless of provenance. The transmission loss allocated to each MW of power flowing in the line is the same, regardless of its provenance. Thus, the loss allocation for each generator that supplies a single outflow line is proportional to the share of its generated output in that line. Proportional sharing follows directly from accepting the Shapley value as the solution concept, with the allocation $\phi(c)$ the same for all players.

4.2.4 Allocation of inflows to outflows: The logic employed in defining the game presented above applies in the context of the network segment shown in Fig. 1. If there was more than one outflow line, how are the inflows to be distributed among the outflows lines from the point of view of sharing out total transmission loss? The essential point is that since $c(q)$ is convex, attributing a disproportionate share of the power flow in a line that carries more power than the others to one generator (or load, if the loss is allocated to loads) will attribute to it a much larger share of transmission losses; that will be unfair.

From an accounting point-of-view, one may consider the outflow from the node to the different lines to be toted-up MW by MW in some order, for example, in the order in which the units of power have been labelled, 1 through R. We need make no assumptions about the precise nature of this disbursement procedure. For instance, one convention might be that successively numbered MWs are accounted to be fed to different lines, until all inflows have been disbursed. Another might be that successively numbered MWs are accounted to be fed through in blocks to a line until its power flow 'target' is met, before the next line is fed, and so on, until all inflows are accounted. Given the accounting procedure, the Shapley value allocation is based on equal consideration of all possible permutations of the set $\{1, \ldots, R\}$. If each $\pi \in \Pi_R$ has the same probability ($1/R!$), this implies that each MW has equal probability of being allocated to any of the outflow lines. In other words, the proportional sharing rule is implicit in the determination of the Shapley value of this cost allocation game.

The assumption that the set of players is a finite set – of units of power (e.g. MW) – can be relaxed without loss. If player size goes to zero, the non-trivial generalisation of the Shapley value to atomic games by Aumann and Shapley [14] preserves the validity of the proportional sharing principle. Cost allocation over the whole network follows in a straightforward additive way from cost allocation in each of the lines in the network. Thus, the proportional sharing rule extends to the entire game.

5 Information theoretic rationale of proportional sharing

A complementary rationalisation of the proportional sharing principle can be provided using information theory. The maximum entropy principle provides a powerful and general method that has been used for solving partial and incomplete information problems when it is important to seek a basis for reasoning in logically indeterminate situations. It is applicable whenever one wants to estimate a unique vector of proportions from significantly incomplete data which could be fitted with many different vectors. Maximum entropy selects the simplest possible result, containing the minimum of structure needed to fit the data. The principle yields the 'best' conclusion possible based on the data at hand; maximum entropy is not merely one regularisation method, but the only consistent procedure. If certain axioms are to be satisfied in the selection procedure, the choice of the probability distribution must be made on the basis of maximum entropy [15]. The principle, as a criterion of choice, recommends that from the set of all probability distributions compatible with all the constraints, we choose the one that maximises entropy. Such a probability distribution will ignore no possibility, but be the minimally structured [15, 16].

Let $\bar{p} = (p_l \ldots p_m)$ be a finite probability distribution, i.e. m real numbers satisfying

$$p_k \geq 0,\ (k = 1, \ldots m);\ \sum_{k=1}^{m} p_k = 1 \tag{17}$$

where p_k represents the probability of the kth possible value taken on by a finite discrete random variable. If there is no other information, the probability distribution that maximises entropy is the uniform distribution. This distribution that adds minimal structure is $p_1 = p_2 = \ldots = p_m = 1/m$.

Application of the principle to the justification of the proportional sharing principle is quite straightforward. Let us consider again the problem of allocation of inflows to outflows (Fig. 1). As before, we will treat each of the inflows and outflows as a stream of successively numbered MWs. The question is therefore what is the probability p_k that the kth incoming MW will go to either line i–m or i–l? In the absence of any additional information, the entropy will be maximised if the probability distribution is the uniform one; the probabilities are the same. If each MW has the same probability of flowing to either i–m or i–l, the node is a perfect 'mixer'. This is consistent with the proportional sharing principle.

6 Conclusions

The tracing methodology is based on the assumption that, at any network node, the incoming flows are proportionally distributed among the outcoming flows. This assumption can be neither proved nor disproved physically and the aim of this paper was to rationalise it. The analysis was performed using as an example the loss allocation problem. It was shown that the proportionality assumption leads to the cost allocation which is aggregation invariant. The proportional sharing principle was rationalised using the game theory and the information theory. We showed that the Shapley value, the solution concept which satisfies all properties one may demand of a loss allocation scheme, substantiates the proportional sharing rule. We have also shown that the rule can be derived from the maximum entropy principle.

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