# STATE-SPACE TECHNIQUE FOR MINIMAL REALISATION OF ANALOGUE CIRCUITS AND SYSTEMS

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## ABSTRACT

An analogue circuit's behaviour can be represented by a number of natural modes. To ensure the compliance of the circuit to a prescribed specification, it is important to test the circuit against the performance of its natural modes. Unspecified mode's characteristics, which could be due to disturbances in the form of faults (i.e. hard or soft), can indicate abnormality in circuits specified behaviour. The work presented here is twofold: (i) an investigation of systems dynamics to estimate the reliability of the circuit when operating in the presence of faults; (ii) a proposed method for minimal realisation.

## 1. INTRODUCTION

The functions of an analogue system are complex, not capable of being adequately described in simple terms. The great variety of possible input and output signals, and the number of possible circuit configurations in which an analogue system may be used make it extremely difficult to determine which of the many system parameters are important. Furthermore, accurate analogue simulation has to be performed at the device level and this is very demanding of CPU time and memory. Consequently it can only practically be applied to small analogue designs, otherwise the analysis effort and time requirements would be prohibitive.

There are numerous techniques in the literature for testing analogue/mixed-analogue VLSI circuits, some of these techniques use simulation methods [1] and others use symbolic methods [2], but few attempts have been made to estimate the robustnace of these circuits to faults (for instance they may identify faults which are not important for system behaviour). Techniques which are either aimed at testing [3] or at identifying faults which have no effect on analogue circuits' behaviour [4], or aimed at testability analysis [5] generally have high computational cost even for relatively small circuits because of the number of variables and parameters involved. Faults which have no effect on a circuit's behaviour are an indication of a redundancy in the circuit [6] which is another contributor to the problem suffered by established techniques since the inclusion of a redundant part of a circuit in any analysis process will add to the overall cost in terms of time and effort. The redundancy may be the result of an accidental failure to implement minimal design in which case there is a need to eliminate such redundancy (i.e. to achieve minimal realisation); or it may be included deliberately in order to satisfy some other design criterion, in which case it is important to be identified and excluded during testing or analysis of the design.

Based on the work presented in [7], this paper is to investigate the effect of faults on the dynamic characteristics of VLSI circuits and their tolerance to such faults, and to propose a state-space based method of evaluating a minimal realisation.

## 2. STATE-SPACE: MATHEMATICAL BACKGROUND

The *state space* representation [6] of a system is given as follows:

$x'(t) = Ax(t) + Bu(t)$ (1)

$y(t) = Cx(t) + Du(t)$ (2)

where,

**x(t):** state vector (n elements); **y(t):** output vector (m elements); **u(t):** input vector (r elements)

**A:** system-interconnection matrix (n.n); **B:** driving matrix (n.r); **C:** output matrix (n.m);

**D:** transmission matrix (m.r)

The diagonal matrix of A (i.e. $A_d$) has its main diagonal representing the distinct poles of the system which can be real or complex ($\lambda_i$; i: 1, 2, ....., n).

$A_d = (\lambda_{ij})$, (3)

where $\lambda_{ij} = 0$ when $i \neq j$.

The state space vector representation of $A_d$ is given by

$d' = A_d\, d$ (4)

(d: diagonal-system vector)

With the natural modes being $e^{\lambda_{ij} t}$

The relationship between the actual states of the system and the diagonal states is given by

$x = V\, d$ (5)

(V: a matrix with eigenvectors as columns).

Differentiating both sides

$x' = V\, d'$ (6)

The matrix V can be considered as a transducer matrix transforming the state vector d into the state vector x. The eignvector can be determined using the following equation

$A\, V_w = \lambda_w V_w$ (7)

where,

$V_w$: an eignvector of A

$\lambda_w$: the corresponding eignvalue

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For undriven system, equation (1) becomes

$x' = A x$ (8)

From (4), (6) and (8)

$$A = V A_d V^{-1} \quad (9)$$

From equation (9), since V can be chosen arbitrarily, there is an infinite number of systems can be obtained, all having the same eigenvalues.

From the above it is informative that systems with different realisations may have the same natural modes, and hence their characteristics are identical. With reference to a reducible system, faults occurring in such a system may result in reconfiguring the system into its reducible realisation without effecting its natural modes (i.e. there is a nonsingular matrix such that $A = V\hat{A}V^{-1}$; $\hat{A}$: faulty system).

## 3. FAULTS AND EIGNVALUES

Consider a faulty system such that the fault introduced has no effect on the output.

From equations (1) and (2)

$$x' + \hat{x} = [A + \hat{A}][x' + \hat{x}] + BU$$

$$y = [c_1 \quad c_2][x' + \hat{x}] + DU$$

separating the faulty signals

$$\hat{x} = \hat{A}\hat{x}$$

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

from equation (7)

$$\begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{bmatrix} \begin{bmatrix} \hat{W}_{11} \\ \hat{W}_{21} \end{bmatrix} = \lambda_1 \begin{bmatrix} \hat{W}_{11} \\ \hat{W}_{21} \end{bmatrix} \quad (10)$$

$$\begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{bmatrix} \begin{bmatrix} \hat{W}_{12} \\ \hat{W}_{22} \end{bmatrix} = \lambda_2 \begin{bmatrix} \hat{W}_{12} \\ \hat{W}_{22} \end{bmatrix} \quad (11)$$

from (10)

$$\hat{a}_{11}\hat{W}_{11} + \hat{a}_{12}\hat{W}_{21} = \hat{\lambda}_1 \hat{W}_{11} \quad (12a)$$

and

$$\hat{a}_{21}\hat{W}_{11} + \hat{a}_{22}\hat{W}_{21} = \hat{\lambda}_1 \hat{W}_{21} \quad (12b)$$

from (11)

$$\hat{a}_{11}\hat{W}_{12} + \hat{a}_{12}\hat{W}_{22} = \hat{\lambda}_2 \hat{W}_{12} \quad (13a)$$

and

$$\hat{a}_{21}\hat{W}_{12} + \hat{a}_{22}\hat{W}_{22} = \hat{\lambda}_2 \hat{W}_{22} \quad (13b)$$

assume

$\hat{W}_{11} = \alpha W_{11}$ and $\hat{W}_{21} = \beta W_{21}$

where $\alpha$ and $\beta$ are factors representing the change in the eignvectors $W_{11}$ and $W_{21}$ respectively in the presence of a fault.

Substituting $\hat{W}_{11}$ in equations 12a and 12b, and re-arranging

$$W_{21} / W_{11} = (\alpha/\beta)[(\hat{\lambda}_1 - \hat{a}_{11}) / \hat{a}_{12}] \quad (14a)$$

$$W_{21} / W_{11} = (\alpha/\beta)[\hat{a}_{21} / (\hat{\lambda}_1 - \hat{a}_{22}) \quad (14b)$$

similarly

$$W_{12} / W_{22} = (\gamma/\kappa)[(\hat{\lambda}_2 - \hat{a}_{22}) / \hat{a}_{21}] \quad (15a)$$

$$W_{12} / W_{22} = (\gamma/\kappa)[\hat{a}_{12} / (\hat{\lambda}_2 - \hat{a}_{11})] \quad (15b)$$

where $\gamma$ and $\kappa$ are factors representing the change in the eignvectors $W_{12}$ and $W_{22}$ in the presence of fault. Re-arranging equations 14a and 14b, and differentiating with respect to $\alpha$ or $\beta$

$$0 = \hat{\lambda}_1 - \hat{a}_{11} \quad \Rightarrow \quad \hat{\lambda}_1 = \hat{a}_{11}, \text{ and } \hat{a}_{12} = 0.$$

Re-arranging equation 15a and 15b, and differentiating with respect to $\gamma$ or $\kappa$

$$0 = \hat{\lambda}_2 - \hat{a}_{22} \quad \Rightarrow \quad \hat{\lambda}_2 = \hat{a}_{22}, \quad \text{and } \hat{a}_{21} = 0.$$

Hence

$$\begin{bmatrix} \hat{\lambda}_1 & 0 \\ 0 & \hat{\lambda}_2 \end{bmatrix}$$

The fault introduces a change in the elements of the main diagonal of the matrix which represents the eignvalues of the system. For the case of non-irreducible systems, the introduction of a fault may not effect the overall system behaviour [7] [8].

**Illustration:** The state space representation of a third order system is given as follows:

$$\mathbf{A} = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [0 \quad 1 \quad 2] \quad \mathbf{D} = 0$$

The system realisation is shown in figure 1.

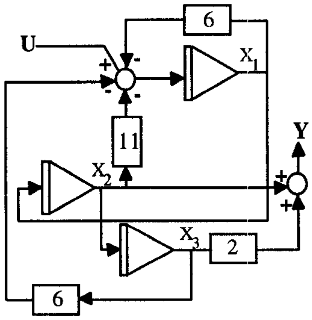


Figure 1 A third order system

$$V=\begin{bmatrix}0.9435 & 0.8729 & 0.5774\\ -0.3145 & -0.4364 & -0.5774\\ 0.1048 & 0.2182 & 0.5774\end{bmatrix}$$

The system response to a unit step input is shown in figure 2.

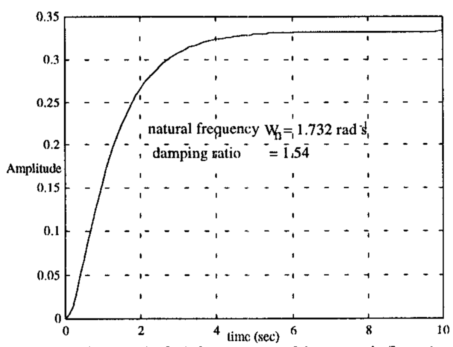


Figure 2 The fault free response of the system in figure 1

A fault is injected into the system such that the output of the integrator of the state variable $x_3$ is kept at 0 value. Hence, the system is reduced to a second order with only two state variables.

From equations 14a and 15b

$$\frac{W_{11}}{W_{21}}=\frac{a_{12}/\beta}{\lambda_1-a_{11}/\alpha} \qquad (16a)$$

$$\frac{W_{12}}{W_{22}}=\frac{a_{12}/\beta}{\lambda_2-a_{11}/\alpha} \qquad (16b)$$

Manipulating 16a and 16b

$\alpha = 3/2, \quad \beta = 2$

re-arranging equations 13a and 13b in terms of the eignvectors ratios, and matching the coefficients with their correspondent in equations 14a and 15b

$\hat{a}_{11} = -4, \ \hat{a}_{12} = -3$

Repeating the above procedure for 14b and 15a

$\hat{a}_{22} = -0, \ \hat{a}_{21} = 1$

The state space representation of the new system (i.e. faulty system) is as follows:

$$\hat{A}=\begin{bmatrix}-4 & -3\\ 1 & 0\end{bmatrix} \quad \hat{B}=\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$\hat{C}=\begin{bmatrix}0 & 1\end{bmatrix} \quad \hat{D}=0$$

$$\hat{V}=\begin{bmatrix}-0.9487 & 0.7071\\ 0.3162 & -0.7071\end{bmatrix}$$

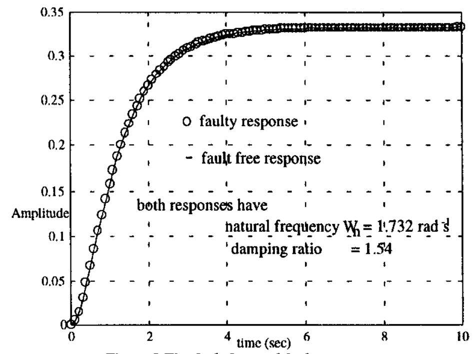


Figure 3 The fault free and faulty response of the system in figure 1

The reduced system realisation (minimal realisation) is shown in figure 4.

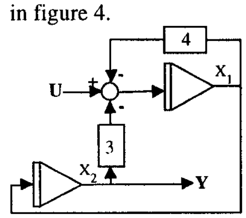


Figure 4 A reduced order system

The eignvector matrix shows that the new system has the same eignvectors ratios and the same eignvalues when compared with the original system.

For the irreducible system in figure 2, a change in any of the eignvalues as a result of a fault(s) would trigger abnormalities in the system's characteristics, as investigated in the following section.

## 4. FAULT INJECTION

To illustrate the effect of hard and soft faults on the system shown in figure 4, only the integrator representing the state variable $x_1$ is considered. The analogue representation of the integrator is shown in figure 5.

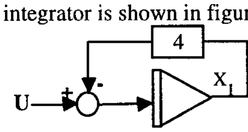


Figure 5 An integrator representing $x_1$

The analogue representation of the integrator in figure 5 is shown in figure 6

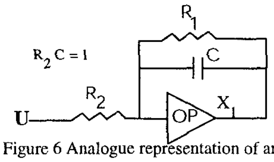


Figure 6 Analogue representation of an integrator

$$\frac{R_1}{R_2} = 4 \quad (17)$$ (the integer '4' is the value of the feedback coefficient in figure 5)

Any slight or large variations (i.e. soft or hard faults) in either $R_1$ or $R_2$ would introduce changes in the value of the feedback coefficient and eventually affect the eignvalue of the state variable $x_1$. The change in the eignvalue yields a change in the system behaviour as illustrated in figure 7.

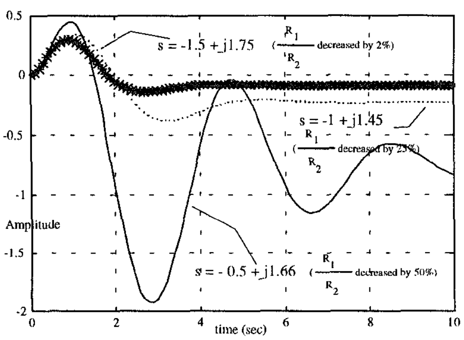


Figure 7 Changes in the behaviour of the system in figure 6 as a result of changes in the coefficient associated with with the state variable $x_1$

With reference to figure 5, It is seen that the characteristics of the poles are severely disturbed by the changes in either $R_1$ or $R_2$. With reference to [6] the changes in the system's parameters are shown below.

| System parameters | R1/R2 decreased by 2% | R1/R2 decreased by 25% | R1/R2 decreased by 50% |
|---|---|---|---|
| natural frequency Wn | 1.732 rad $s^{-1}$ | 1.733 rad $s^{-1}$ | 1.761 rad $s^{-1}$ |
| damped frequency Wd | 1.75 rad $s^{-1}$ | 1.65 rad $s^{-1}$ | 1.69 rad $s^{-1}$ |
| damping ration $\xi$ | 0.65 | 0.29 | 0.28 |
| overshoot Mp | 0.064 | 0.38 | 0.39 |
| tp | 6.4% overshoot occurs at 1.794 sec | 38% overshoot occurs at 1.9 sec | 39% overshoot occurs at 1.85 sec |
| settling time | 2.637 sec | 5.97 sec | 6.08 sec |
| No. of oscillations necessary to reach ts | 0.558 oscillations | 1.576 oscillations | 1.637 oscillations |

The irreducable system obtained above represents an overdamped system, but its behaviour was drifted into oscillation prior to reaching its steady state as a result of the changes in R1 and R2 which in turn affect the state variable $x_1$ (i.e. the characteristics of a natural mode). If the variations in the characteristics of this mode are within a prescribed limit permitted by the specifications, then the system could be kept in operation without a noticeable compromise in performance.

## 5. CONCLUSIONS

Systems are vulnerable to faults. Depending on the realisation of the system under investigation, faults occurring in the system may not have an effect on the system's behaviour. Hence, systems under such faults can operate without compromising performance. The method introduced here uses an investigation of systems' dynamic which is represented by a number of natural modes, and is independent of systems' layout. Unexpected variations in a natural mode's characteristics will trigger abnormalities in systems behaviour. Recording such variations and the time span associated with these variations provides necessary information for identifying faults and estimating the reliability of systems when operating in the presence of faults.

In the presence of a redundancy in a circuit, the paper demonstrated a method which makes use of systems' dynamic related information to eliminate hardware redundancy.

## 6. REFERENCES


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